THE HETEROMORPHISM IN CATEGORY THEORY

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ABSTRACT

The Heteromorphism in Category Theory

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To many mathematicians, category theory is simply a very general, abstract language for the high-level organization of concepts. From this perspective, categories are nothing more than gossamer webs floating in the Platonic breeze, unconcerned with the trials and tribulations of concrete mathematics, only helpful when sheer complexity necessitates a transcendent viewpoint. This is why the subject is still largely unknown, unused, and unexplored by the mathematical community.

Category theorists, of course, tell a different story. Most wholeheartedly agree that its potential to elucidate and unify mathematics is greatly underestimated. This is not a grand hope of an idealistic rebellion, but a deep intuition developed through devotion to a way of thought whose purity demands the most stringent justification, via absolute conceptual mediation instead of experience.

The category is the pure abstraction from which there are only two paths to meaningful, concrete truth: the study of mathematics, and (mathematical) introspection. There is no fantasy world beyond mathematics, but we do have the crucial capacity to temporarily

transcend our subjects. This removes biases of particular formal systems, and brings everything back into the universal language of objects and arrows. As a result, we coalesce mathematical perspectives and see their concrete interrelation.

At first, these two paths were not strongly related; but as tools have improved, categorical mathematics and introspection have begun to be deeply intertwined. Interesting phenomena in one coincide with the other, and following these insights brings us closer and deeper into the real content of mathematics. Category theorists are thereby becoming intimately familiar with subtle and important truths about mathematics which are imperceptible in other subjects.

Because of this and other developments, category theory is becoming a better candidate for a truly unified approach to mathematics. But the root cause of the disconnect between the two is ironically the assumption that the latter subsumes the former. Category theory, even though it is explicitly the study of conceptual systems, is considered to be one like any other, simply because we do not yet know what other option is possible. This almost defeats the purpose, and basically forces us to treat categories the same way as the objects they are meant to study.

If we can ever hope to unify mathematics, we must characterize the category in a way that accounts for its own self-predication. This inevitably involves interpreting reason as a *creative process* which will change how we conceive mathematics and its relationship to the world. I believe that category theory will culminate into a universal method through which all of mathematics can be theorized and practiced together, no longer separated into different formal systems and only externally related as disparate notions. Moreso, I believe that developing such a method could demonstrate how other subjects of mathematics actually arise from categorical reasoning.

However, we are far from this point for a variety of reasons. We now have a potentially universal language, but precise steps towards this goal are unknown, and the subject is being neglected because people have not seen enough evidence that categorical concepts are useful in the toil of mathematics. These two problems are one and the same. Because of the mathematical conception of category theory, bias toward maintaining its formal purity has caused a neglect of certain ideas which may actually be the key to their interconnection. I will survey what I believe to be the most basic of these: the heteromorphism.

DEDICATION

To my family.

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1. CONCEPTS

A *heteromorphism* is an "arrow between objects of different categories." This cannot be expressed in ordinary theory, because objects and arrows are only defined within a category. Nevertheless, they are ubiquitous for an important reason. While the categorical arrow *preserves* the structure of the category, the heteromorphism *changes* the structure of the object. An arrow in the category of groups is a group homomorphism; a heteromorphism from a set to a group applies a group structure on that set. Which one is fairly trivial, and which is interesting?

While this notion may seem superfluous to category theorists for its formal elusiveness, its central relevance to mathematics cannot be ignored. Mathematical thought is an evolving process which primarily consists in learning about an object by modifying it, and dealing with the complex consequences of those changes. Although heteromorphisms are not categorically definable, they are governed from head to tail by categorical reasoning. For example, it makes sense to generate an algebraic structure only in certain circumstances, to be used for certain purposes; and when we do, the elements of the set are not really indistinguishables, but mathematical objects brought together by a certain reason, and the group structure reflects and elucidates that reason. These and many other "fluid" mathematical ideas are common knowledge, but we have no way to say them precisely.

What if there is a way to characterize such ideas with heteromorphisms? It is the right notion for the question, and it has never been explored. This would be important for several reasons: it would introduce a notion of dynamism into mathematics, and we would begin to more closely see the ethereal boundary between abstract conceptual systems and concrete conceptual actions.

Profunctor

Because of their pervasiveness, recognition of heteromorphisms by category theorists was inevitable, even if they could not be defined in the conventional language. So, an important tool was created, but even though they describe heteromorphisms, they became considered a replacement for their consideration, and so the latter have almost never been discussed. A *profunctor* $C \nrightarrow D$ is a functor $D^{op} \times C \to$ Set, which sends (d, c) to the "set of heteromorphisms from d to c." This is the abstraction of the generalization from a function to a relation; as this is a very important distinction in conventional mathematics, it is similarly fundamental in category theory.

Hom Functor

As soon as category theorists started to formalize the basics, something interesting arose - although it was ignored at the time, because there was mainly consideration of only the basic formal elements of category theory. We needed a tool to describe the collections of arrows between objects, called *hom-sets*. But if one wants to be able to both precompose and postcompose these arrows with others, a simple duality occurs. Suppose we have $a \to b \to c \to d$ and consider (b, c). To get (b, d), just postcompose in the second argument. But to precompose $(b, c) \to (a, c)$ we see that the arrow is *reversed* in the first argument. Thus, this parameter of the hom-functor must actual be the *opposite* category, the category with all arrows reversed:

$$\text{Hom} : C^{op} \times C \to \text{Set}$$

We can see that this is actually a profunctor $C \nrightarrow C$, and in fact it is known as the "identity profunctor." We find that when we want to properly formalize the arrows of a category, they attain a double meaning as both homo-morphisms and heteromorphisms.

Presheaf

A *presheaf* is a functor $H : C^{op} \to$ Set. This is a central organizing and communicating tool in category theory, especially in *topos theory*, the vast generalization of set theory to study the much more subtle and complex notions of "element" that arise in categories. A presheaf is *representable* if it is isomorphic to a functor $C(-, c)$ for some object c in C.

From a heteromorphic perspective, a presheaf is simply a profunctor $H : 1 \nrightarrow C$, where 1 is the terminal category. In an arbitrary category, an arrow from the terminal object $* \to a$ is called a *global element* of a, generalizing the notion of object in Cat. Identifying an object with its representable presheaf $C(-, c)$, the notion of a "profunctor global element" generalizes this to any presheaf. Similarly, a *copresheaf* $K : C \to$ Set is a profunctor $C \nrightarrow 1$, or a functor $1^{op} \times C \to$ Set, sending $(*, c)$ to its set of heteromorphisms. This generalizes the representable copresheaf $C(c, -)$ to any copresheaf.

We can see an interesting symmetry here: while globally the profunctor is $1 \nrightarrow C$, this is the functor $C^{op} \times 1 \to$ Set, which sends $(c, *)$ to the set of heteromorphisms from c to $*$. In the same way that every category has a unique arrow to the terminal category, every object has a set of heteromorphisms to the "terminal object." This interpretation clarifies the Yoneda embedding as the "free cocompletion," and the co-Yoneda Lemma that every presheaf is a colimit of representables:

Universality

The most striking evidence for the importance of heteromorphisms can be found in the central notion of universality. Basically, this property is the reason why the language of category theory is "universal," why objects and arrows can ideally characterize any mathematical concept. All of the primary tools used in category theory, such as co-limit, co-end, extension-lift, and related notions such as representability and adjunction, are so essential to the subject precisely because they exhibit this universality.

Saunders Mac Lane, a founder of category theory, once said: "Good general theory does not search for the maximum generality, but for the right generality." Universality epitomizes this idea, by characterizing "ideal forms" of concepts. For example, the free group on a set is the "most group-like" form of that set, and as we know, there is a homomorphism from it to any group made from that set. Or when we define bilinear functions and need a product to accomodate this, we take the usual direct product and then modulate by the minimal equivalence relation to ensure bilinearity; the tensor product is the "most bilinear" product, such that any bilinear function from their product to another module factors through the tensor product. This distinguishing of idealities is an indispensable guiding principle in an era of abstract mathematics that generates hundreds of new concepts every day.

Semantically, "universality" it just as much about individuality. The "ideality" is properly characterized by *unique* factorization. For example, the product of two objects $a \times b$ is such that any pair $c \to a$, $c \to b$ factors uniquely through the product; and this completely describes what is important about $a \times b$, in that it is the unique object which "knows" everything about the pair (a, b) - and nothing else. This individuality is the essential ingredient in concrete categorical truth. We can gain perspective from great generality and study global phenomena, but the true use of mathematics in life is understanding the idiosyncracies of individual objects. Each exhibits many different structural properties, and can thus be placed in many contexts for study, but its uniqueness as a mathematical object is precisely what makes it real and important.

Below are the different ways of describing universality [1]. The left, universal arrow, and the right, universal element, can be seen to be unified by the middle, the relatively neglected characterization by representable profunctors, when $C(S-, c) = H : C \nrightarrow D$ or $C(c, S-) = H : D \nrightarrow C$. Rather than merely an element of a set, the *universal element* η_c or ε_c can be more vividly seen as precisely the universally factorizing heteromorphism.

I believe that its transparency and simplicity could help unify our understanding of the universal constructions and related notions - and by association, category theory as a whole.

Arrow **Heteromorphism** **Element**

$$C \xleftarrow{S} D$$

$$\begin{tikzcd} Sd \arrow[r, "f'"] \arrow[d, "Sf"'] & c \arrow[d, equal] & d \arrow[d, "\exists!\bar{f}"'] \\ Sr \arrow[r, "u"'] & c & r \end{tikzcd}$$

S to c

$$C \relbar\joinrel\mapstochar\joinrel\rightarrow D$$

$$\begin{tikzcd} & d \arrow[dl, dashed, bend right, "\varphi"'] \arrow[d, "\bar{f}"] \\ c & r \arrow[l, dashed, "\varepsilon_c"] \end{tikzcd}$$

Right Representation

$$D \xrightarrow{H} \text{Set}$$

$$\begin{tikzcd} d \arrow[d, "\exists!\bar{f}"] & * \arrow[r, "\{\varphi\}"] \arrow[d, equal] & Hd \\ r & * \arrow[r, "\{\varepsilon_c\}"'] & Hr \arrow[u, "Hf"'] \end{tikzcd}$$

Contravariant

$$C \xleftarrow{S} D$$

$$\begin{tikzcd} Sr \arrow[d, "Sf"'] & c \arrow[l, "f'"'] \arrow[d, equal] & r \arrow[d, "\exists!\underline{f}"'] \\ Sd & c \arrow[l, "v"] & d \end{tikzcd}$$

c to S

$$D \relbar\joinrel\mapstochar\joinrel\rightarrow C$$

$$\begin{tikzcd} r \arrow[d, "\underline{f}"'] & c \arrow[l, dashed, "\eta_c"'] \arrow[dl, dashed, bend left, "\varphi"] \\ d & \end{tikzcd}$$

Left Representation

$$D \xrightarrow{H} \text{Set}$$

$$\begin{tikzcd} r \arrow[d, "\exists!\underline{f}"] & * \arrow[r, "\{\eta_c\}"] \arrow[d, equal] & Hr \arrow[d, "Hf"] \\ d & * \arrow[r, "\{\varphi\}"'] & Hd \end{tikzcd}$$

Covariant

Adjointness

Intuitively, adjointness is the “closest approximation of an inverse” when one does not necessarily exist. Because category theory studies the fine relations of concepts, it is not surprising that the notion of an “irreducible unit of difference” is very important. In fact, some category theorists regard adjointness as the central concept, and generally hold that they are the source of everything nontrivial and interesting in category theory.

There are several different ways of defining when two functors are adjoint. The characterization most useful for everyday category theory is the hom-isomorphism:

$$\begin{tikzcd} A \arrow[r, bend left, "F"] & B \arrow[l, bend left, "G"] \end{tikzcd}$$

$$A(a, Gb) \cong B(Fa, b)$$

This looks familiar because it is precisely the double-sided version of universality. Functors can be seen more generally as representable profunctors, where each $H : C \to D$ induces two, $D(1, H) : C \nrightarrow D$ and $D(H, 1) : D \nrightarrow C$. So, adjointness is when the two profunctors $A(1, G), B(F, 1) : B \nrightarrow A$ are equal. This leads to a simple heteromorphic characterization: functors are adjoint when their induced heteromorphisms have both a left and right representation. [2]

$$\begin{tikzcd} a \arrow[r, "\exists! g"] \arrow[d, dashed, "\eta_a"'] \arrow[dr, dashed] & Gb \arrow[d, dashed, "\epsilon_b"] \\ Fa \arrow[r, "\exists! f"'] & b \end{tikzcd}$$

This perspective describes the hom-isomorphism simply and visually, and also unifies the other characterizations: below, the top and bottom squares are the universal factorization by unit and counit, and interestingly, the left and right squares are a new form of the "triangle identities," via the heteromorphic Yoneda Lemma (discussed later). It also, of course, incorporates the characterization by "cograph of a profunctor."

$$
\begin{tikzcd}
& GFa \arrow[r, "GFg"] \arrow[dr, "Gf"] & GFGb \arrow[dr, dash, bend left] & \\
GFa \arrow[d, dashed, "\eta_{GFa}"'] \arrow[dr, dashed, "\varepsilon_{Fa}"'] \arrow[ur, dash, bend left] & a \arrow[l, "\eta_a"'] \arrow[u, "\eta_a"] \arrow[r, "\exists! g"] \arrow[d, dashed, "\eta_a"'] \arrow[dr, dashed, "\varphi"] & Gb \arrow[u, "\eta_{Gb}"'] \arrow[r, "\eta_{Gb}"] \arrow[d, dashed, "\varepsilon_b"] \arrow[dr, dashed, "\eta_{Gb}"] & GFGb \arrow[d, dashed, "\varepsilon_{FGb}"] \\
FGFa \arrow[r, "\varepsilon_{Fa}"'] \arrow[dr, dash, bend right] & Fa \arrow[r, "\exists! f"'] \arrow[dr, "Fg"] & b & FGb \arrow[l, "\varepsilon_b"'] \arrow[dl, dash, bend left] \\
& FGFa \arrow[u, "\varepsilon_{Fa}"] \arrow[r, "FGf"'] & FGb \arrow[u, "\varepsilon_b"'] &
\end{tikzcd}
$$

This is an example in which the heteromorphism can help unify our understanding of central notions of category theory. Additionally, this perspective is by default *local* rather than global, because we specify heteromorphisms objectwise, then we may further universally quantify to recover global definitions.

Adjointness is an ideal example of the deep insight of categorical thought: when you try to describe this basic notion of difference, it is the simplest asymmetry: in an adjoint pair, one is a "left inverse from below" and a "right inverse from above," but the two are quite distinct, and functors are usually only one or the other, but they are most of all *uniquely* and *universally* determined.

The oscillations that these produce in each category are called co-monads, and they are fundamental to category theory, yet conceptually distinct from the complementary notion of adjoint. Most interesting is that there is even an adjunction between monads and adjunctions themselves, called the semantics-structure adjunction. Simple specifications of the adjunction, such as laxness or idempotence, are also immense sources of very efficient and ideal information. One intuitive way of describing adjoints is simply "the most efficient solution to a problem." For an example of their concrete reality, every two-dimensional topological quantum field theory is equivalent to a Frobenius monad, which is induced by an *ambidextrous adjunction*, which is an adjoint triple $F \dashv G \dashv H$ such that $F \cong H$.

Lastly, I believe that heteromorphic Kan extensions, along with the aforementioned triangle identities, can lead to a generalization of heteromorphic adjoints to the "elementary" adjunction in any bicategory, rather than just Cat. This is seen as more fundamental by many category theorists, because the definition does not rely on Set, and is much more general in practice. I do not think there will be any problem with this generalization, and it will be insightful because most important categorical notions, like monads or enriched constructions, are naturally bicategorical.

Yoneda Lemma

Besides simplicity and transparency, heteromorphisms also elucidate another important phenomenon: the transfer of information between different levels of abstraction. The Yoneda Lemma is a central theorem in category, which describes the natural relationship between representable functors and presheaves.

Lemma. *(Yoneda) If $K : D \to$ Set is a functor from D and r an object in D (for D a category with small hom-sets), there is a bijection which sends each natural transformation $\alpha : D(r, -) \to K$ to $\alpha_r 1_r$, the image of the identity $r \to r$.*

$$y : Nat(D(r, -), K) \cong Kr$$

$$\begin{tikzcd}
D(r,r) \arrow[r, "\alpha_r"] \arrow[d, "D(r,f)"'] & K(r) \arrow[d, "K(f)"] & r \arrow[d, "f"] \\
D(r,d) \arrow[r, "\alpha_d"] & K(d) & d
\end{tikzcd}$$

Here, what appeared as only a bijection of sets can be seen to be a direct result of composition of heteromorphisms, by the objectwise components of the natural transformation. Here we have the simple idea that *identity*, as a heteromorphism, is both the left and right universal element of its representable functors, and this induces a correspondence to arbitrary presheaves. We can also see a little more clearly the significant difference between covariance and contravariance.

$$
\begin{tikzcd}
r \arrow[d, dashed, "1_r"'] \arrow[dr, dashed, bend left, "f"] & \\
r \arrow[r, "\exists! f"'] \arrow[d, dashed, "\alpha_r"'] & d \arrow[d, dashed, "\alpha_d"] \\
Hr \arrow[r, "Hf"'] & Hd
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
d \arrow[r, "\exists! f"] \arrow[dr, dashed, bend right, "f"'] & r \arrow[d, dashed, "1_r"] \arrow[dr, dashed, bend left, "f^{op}"] & \\
& r \arrow[r, "\exists! f^{op}"'] \arrow[d, dashed, "\alpha_r"'] & d \arrow[d, dashed, "\alpha_d"] \\
& Hr \arrow[r, "Hf"'] & Hd
\end{tikzcd}
$$

Covariant *Contravariant*

Isbell Duality

Even the innocent Hom-functor, when viewed as the identity profunctor, can offer deep insight. An *enriched category* is a category whose hom-sets are generalized to objects in a suitable enriching category V (generally required to be cocomplete, symmetric monoidal). Accordingly, profunctors generalize to enriched profunctors - rather than a set of heteromorphisms, we could have a topological space, or a vector space, or an actual category of heteromorphisms; I believe that this notion, because of its apparent contradiction of heteromorphism, is significant and deserves closer inspection. We then have the following idea.

Theorem. *(Isbell) Suppose C, D are categories enriched over a co-complete symmetric monoidal category V. Then a V-enriched profunctor $F : D^{op} \otimes C \to V$ induces the adjunction:*

$$(V^C)^{op}(F^* p, q) \cong V^{D^{op}}(p, F_* q)$$

In the case that F is Hom : $C^{op} \otimes C \to V$, this is a central theorem known as *Isbell Duality*. This is known as the duality between algebra and geometry, because higher algebra deals with presheaves, while higher geometry deals with copresheaves. This is an example of a modern discovery whose importance cannot yet be estimated.

Formalization

Incorporating heteromorphisms as a basic notion of category theory could even clarify its definition. Currently, the category assumes the notion of "collection," which is just a generalized notion of set to accommodate for size issues. But, in the "universal language" of category theory, a collection is supposed to be a discrete category. But these two definitions are blatantly circular, so they cannot both be true; moreover, if we rely on sets in the very definition of a category, we will still be implicitly limited by set-theoretic assumptions.

Roughly, I think that there may be a way to identify a category with its identity-profunctor, and derive identity and associativity of composition simply from the properties of the Yoneda embedding. Of course, notions such as profunctor will have to be understood as more primitive than they are presently, which I believe will be natural and helpful in moving toward unification with logic and type theory. The notion of a set may turn out to be unavoidable in the definition, but it must at least be avoided as more primitive than the category. And in the process of trying to accommodate for the heteromorphism as primitive, I believe that certain important categorical notions, such as those surveyed here, could be elucidated as involved in the definition of category.

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, dashed, "1_a"'] \arrow[dr, dashed, "f"] & b \arrow[d, dashed, "1_b"] & & \\
a \arrow[r, "f"'] & b \arrow[r, "g"] \arrow[d, dashed, "1_b"'] \arrow[dr, dashed, "g"] & c \arrow[d, dashed, "1_c"] & \\
& b \arrow[r, "g"'] & c \arrow[r, "h"] \arrow[d, dashed, "1_c"'] \arrow[dr, dashed, "h"] & d \arrow[d, dashed, "1_d"] \\
& & c \arrow[r, "h"'] & d
\end{tikzcd}
$$

Associativity of Composition

Co-limit

The co-limit is a simple but important version of universality, which helps illustrate the naturality and utility of heteromorphisms even in the basic and common constructions. Limits and colimits essentially “encapsulate the information” of a (diagram) category in two symmetric ways. The natural transformation definition of co-limit can also be seen as a way to formalize the heteromorphisms between the diagram category and the ambient category - and this is more precisely how we actually picture a “universal cone.”

Kan Extension, Lift

Kan extensions are very general and useful constructs, which subsume co-limit and every universal construction in some sense. For example, an adjunction is a Kan extension or lift of an identity morphism, which gives you the notion of “closest approximation of inverse.” Below one can see from the heteromorphic perspective that they create cyclic commutative diagrams.

$$
\begin{tikzcd}
A \arrow[rr, bend left, "g"] \arrow[dr, "f"] \arrow[dr, bend left] \arrow[dr, bend right] & & C \\
& B \arrow[ur, "h"'] \arrow[ur, bend left] \arrow[ur, bend right] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{rift}_h g \arrow[dr, dashed, bend right] & f \arrow[l, "\exists!\bar{\alpha}"'] \arrow[d, dashed] \arrow[dr, dashed, bend left] & & h \arrow[r, "\exists!\bar{\beta}"] \arrow[d, dashed] \arrow[dl, dashed, bend right] & \mathrm{ran}_f g \arrow[dl, dashed, bend left] \\
& g \arrow[r] \arrow[d, dashed] \arrow[dl, dashed, bend right] & hf \arrow[r] \arrow[dl, dashed, bend left] \arrow[dr, dashed, bend right] & g \arrow[d, dashed] \arrow[dr, dashed, bend left] & \\
\mathrm{lan}_f g \arrow[r, "\exists!\underline{\beta}"'] & h & & f & \mathrm{lift}_h g \arrow[l, "\exists!\underline{\alpha}"]
\end{tikzcd}
$$

Here, the heteromorphisms are natural transformations (2-morphisms) between functors (morphisms) of different functor categories. One can derive the usual properties by substituting morphisms for the variables, such as the extensions themselves, etc.

Currently, the only kind of Kan extension which is mathematically significant is *pointwise*, meaning that it is preserved by representable functors, and the reason is unknown. I believe that a closer investigation of Kan extensions incorporating heteromorphisms may be able to tell why this is so - and possibly even a theorem that “all” extensions are pointwise.

Co-end

This universal construction to me seems quite pertinent to this survey, but it is not yet clear to me how the heteromorphism characterization fits here. It seems something like a “co-product of identity morphisms” rather than the objects themselves, which provides the fundamental idea of universal dinaturality. [3]

Weighted Co-limit

The weighted colimit is arguably the most universal of universal constructions, and I expect to see an interesting phenomenon in studying its generalization from the "Hom-weighted co-end" with the use of heteromorphisms.

Sheaf

A *sheaf* is a presheaf which fulfills a condition about the accordance of "local" and "global" information. Given the above characterization of presheaves as evaluating objects as heteromorphisms to the terminal category, this definition seems to be exactly what one would need to keep that interpretation consistent. Since a topos is a category of sheaves over a site, I believe this could give a simpler understanding of the fundamental nature of the topos.

Comma Object

generalized element, global element, generalized universal bundle, loop space object

Monad

algebra over a monad/endofunctor/profunctor, monadic adjunction, bar resolution

Fibration

pertaining to the notion of unique factorization

Co-power, Hom-Tensor

This fundamental multivariable adjunction, involving the simplest weighted co-limits and a unique and central adjunction, I believe will also have a deep heteromorphic interpretation. The tensor product is the example for which Mac Lane had to switch from universal arrow to universal element, because as binary it is not a left adjoint.

Ambimorphic Duality

dualizing object, concrete duality, intersection type, star-autonomy

Proarrow Equipment

A *proarrow equipment* on a bicategory freely adds “forgetful” right adjoints to every 1-morphism. The canonical equipment is Prof, relative to Cat. Equipments and related concepts such as *virtual double category* and *generalized multicategory* are fundamental to the notions of enrichment and internalization, and more generally the idea of “formal category theory.” [4] While speculative, it seems intuitive that each category, as the aforementioned “primitive" identity profunctor in which morphisms are not assumed determinate, gives its own canonical proarrow equipment:

$$(1, f) \dashv (f, 1)$$

$$\begin{tikzcd}
a \arrow[r, "\exists! f"] \arrow[d, dashed, "1_a"'] \arrow[dr, dashed, "f"] & b \arrow[d, dashed, "1_b"] \\
a \arrow[r, "\exists! f"'] & b
\end{tikzcd}$$

Equipment

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